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STARTING SOLUTION FOR AN ABLATING HOLLOW CYLINDER

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Abstract—The "embedding technique", developed by Boley is applied to a hollow cylinder with an ablating inner boundary to obtain a general starting and short time solution. Numerical results are obtained giving the motion of the ablating boundary and the temperature distribution in the vicinity of the moving boundary for ice and lead.

1. INTRODUCTION

A CLASS of heat conduction problems in which the medium changes its phase at a specific temperature with emission or absorption of heat has gained considerable importance in the last decade.

The first exact solution to this class of problems, employing the similarity method, was given in 1860 [1]. Springer and Olson [2, 3] obtained numerical solutions for problems involving axisymmetric solidification and melting of a medium bounded by two concentric cylinders of finite length.

Boley [4] introduced the "embedding technique" for such problems. The actual body, in which the temperature distribution and the position of the transition region between the two phases have to be determined, is considered as a part of a fictitious body which envelops it completely. The heat flux at the surface of the fictitious body is determined from the specified conditions on the transition surface of the actual body. This allows one to assign a simple geometry to the fictitious body, even though the actual body may be quite complicated, and simplifies the formulation of the problem. This method was also used to solve some problems with moving boundaries for finite regions, such as slabs subjected to various boundary conditions [5, 6]. Reference [5] has a comprehensive bibliography for such problems.

In the present paper the embedding technique, which was used by Boley [6] to obtain a starting solution for a finite slab, is applied to a hollow cylinder with an ablating inner surface, in order to obtain the starting solution. An example for such a problem is a solid-fuel rocket where the fuel core is burning from the inside out. The solution is given for the axisymmetric case with a constant heat input up to the melting time at the inner boundary, the outer boundary being kept at ambient temperature. The problem is formulated in terms of dimensionless parameters, so that the temperature distribution and the position of the moving boundary of the cylinder can be obtained for a large class of materials.

2. THE FORMULATION OF THE PROBLEM

Consider a hollow cylinder (Fig. 1) of inner radius a and outer radius b, subject to the boundary conditions

$$-k\frac{\partial T_b}{\partial r} = Q_b \quad \text{at} \quad r = a \qquad 0 < t \leq t_m$$
$$T_b = 0 \quad \text{at} \quad r = b \qquad 0 < t \leq t_m \tag{1}$$

where k is a constant. $T_b(r, t)$ is the temperature (before melting) and Q_b is the prescribed heat input. Equation (1) together with a prescribed initial temperature (here taken as zero), and with the heat conduction equation

$$K\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)T_b = \frac{\partial T_b}{\partial t} \qquad \begin{array}{l} a \leq r \leq b \\ 0 < t \leq t_m \end{array} \tag{2}$$

give a complete formulation of the problem up to the time of melting.

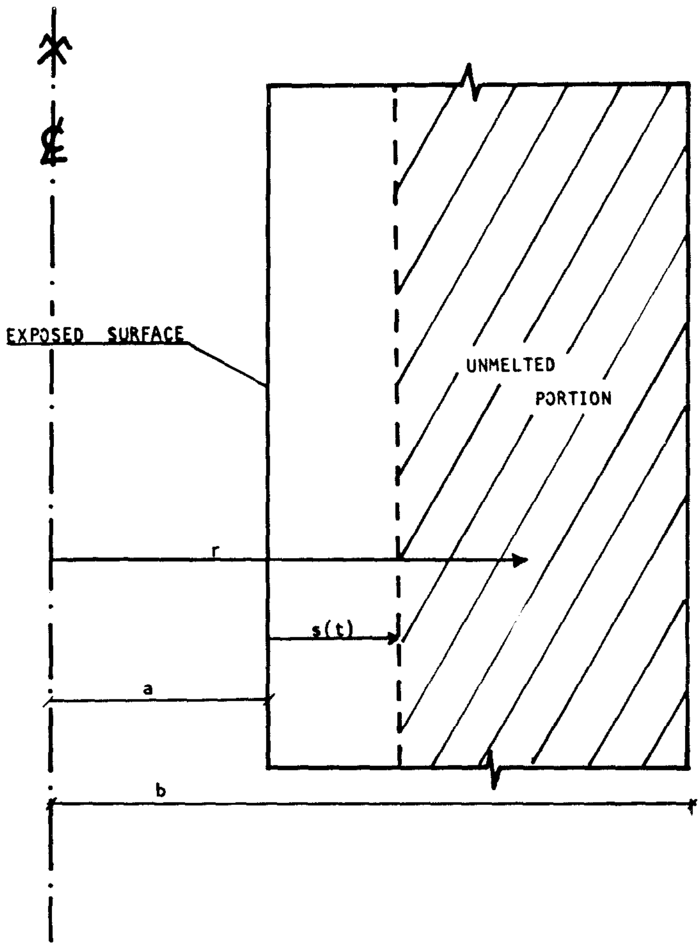

FIG. 1. Cross section of a hollow cylinder with a moving inner boundary.

Let t_m be the time at which the inner surface temperature reaches the melting temperature T_m, so that

$$T_b(a, t_m) = T_m.$$

In terms of dimensionless quantities, the solution of equation (2) can be written as

$$\theta_b = -\frac{q_b}{m}\left[-\ln\frac{R}{\beta}+\pi\sum_{i=1}^{\infty}\frac{1}{\lambda_i}\frac{J_0^2(\lambda_i\beta)}{[J_1^2(\lambda_i)-J_0^2(\lambda_i\beta)]}[J_0(\lambda_i R)Y_1(\lambda_i)-J_1(\lambda_i)Y_0(\lambda_i R)]e^{-K_d\lambda_i^2(1+y)}\right] \quad (3)$$

where

$$\theta_b = \frac{T_b}{T_m}, \qquad R = \frac{r}{a}, \qquad \beta = \frac{b}{a}, \qquad K_d = \frac{K}{a^2}t_m, \qquad y = \frac{t}{t_m}-1$$

$$m = \frac{kT_m}{Q_0 a}, \qquad Q_0 = \frac{T_m k\sqrt{\pi}}{K^{\frac{1}{2}}t_m^{\frac{1}{2}}}, \qquad q_b = \frac{Q_b}{Q_0}$$

and J_0, J_1, Y_0, Y_1 are Bessel functions of the first and second order and first and second kind respectively and λ_i are the roots of the equation

$$J_1(\lambda_i)Y_0(\lambda_i\beta)-J_0(\lambda_i\beta)Y_1(\lambda_i) = 0$$

Equation (3) can be expanded in a power series which is convenient for use in the computations after melting:

$$\theta_b = \sum_{n=0}^{\infty}\sum_{i=0}^{n} a_{ni}y^i(R-1)^{n-i} \quad (4)$$

where

$$a_{ni} = \frac{1}{i!(n-i)!}\left[\frac{\partial^n\theta_b}{\partial y^i\partial R^{n-i}}\right]_{\substack{R=1\\y=0}}$$

$$a_{00} = \theta_b(r, y)\Big|_{\substack{R=1\\y=0}} = 1$$

$$a_{10} = -\frac{q_b}{m}$$

$$a_{11} = 2q_b\frac{K_d}{m}\sum_{i=1}^{\infty}\frac{J_0^2(\lambda_i\beta)}{J_1^2(\lambda_i)-J_0^2(\lambda_i\beta)}e^{-k_d\lambda_i^2}$$

$$a_{20} = \frac{1}{2}\frac{q_b}{m}\left[1+2\sum_{i=1}^{\infty}\frac{J_0^2(\lambda_i\beta)}{J_1^2(\lambda_i)-J_0^2(\lambda_i\beta)}e^{-k_d\lambda_i^2}\right]$$

$$a_{21} = 0$$

$$a_{22} = -\frac{q_b}{m}K_d^2\sum_{i=1}^{\infty}\frac{J_0^2(\lambda_i\beta)}{J_1^2(\lambda_i)-J_0^2(\lambda_i\beta)}\lambda_i^2e^{-K_d\lambda_i^2}$$

$$a_{30} = -\frac{q_b}{3m}\left[1+\sum_{i=1}^{\infty}\frac{J_0^2(\lambda_i\beta)}{J_1^2(\lambda_i)-J_0^2(\lambda_i\beta)}e^{-K_d\lambda_i^2}\right]$$

$$a_{31} = -q_b \frac{K_d}{m} \sum_{i=1}^{\infty} \frac{J_0^2(\lambda_i \beta)}{[J_1^2(\lambda_i) - J_0^2(\lambda_i \beta)]} \lambda_i^2 e^{-K_d \lambda_i^2}$$

$$a_{32} = 0$$

$$a_{33} = \frac{q_b}{3} \frac{K_d^3}{m} \sum_{i=1}^{\infty} \frac{J_0^2(\lambda_i \beta)}{[J_1^2(\lambda_i) - J_0^2(\lambda_i \beta)]} \lambda_i^4 e^{-K_d \lambda_i^2}.$$

For $t \geq t_m$ the temperature distribution (denoted as T) must satisfy the equation

$$K\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) T = \frac{\partial T}{\partial t} \tag{5}$$

$$a + s(t) < r < b \qquad t \geq t_m$$

subject to the conditions

$$\begin{aligned} T[a+s(t), t] &= T_m \\ -k\frac{\partial T}{\partial r} &= Q_b - \rho l \frac{\mathrm{d}s}{\mathrm{d}t} \qquad r = a + s(t) \\ T(b, t) &= 0 \\ T_b(r, t_m) &= T(r, t_m). \end{aligned} \tag{6}$$

Where $s(t)$ is the thickness of the ablating region and ρ and l are the density and the latent heat respectively. The actual cylinder of varying thickness is replaced by an equivalent one of the constant thickness $(b-a)$, as shown in Fig. 2.

The heat input on the equivalent cylinder at $r = a$ is $[Q^*(t) + Q'(t)]$ where $Q^*(t)$ is the analytic continuation of the actual heat input Q_b, before melting, into the region $t > t_m$, while $Q'(t)$ is an unknown fictitious function of time. The temperature T^* due to the given $Q^*(t)$ is simply the analytic continuation of the temperature T_b before melting. The temperature T' due to the unknown function $Q'(t)$ can be obtained in the form of a Duhamel integral:

$$T'(r, t) = \int_0^t Q'(t-\tau) \frac{\partial T_0}{\partial \tau}(r, \tau)\, \mathrm{d}\tau$$

where T_0 is the temperature distribution in an infinite region with zero initial temperature, bounded internally by a circular cylinder with radius $r = a$, [1] and is applicable in this

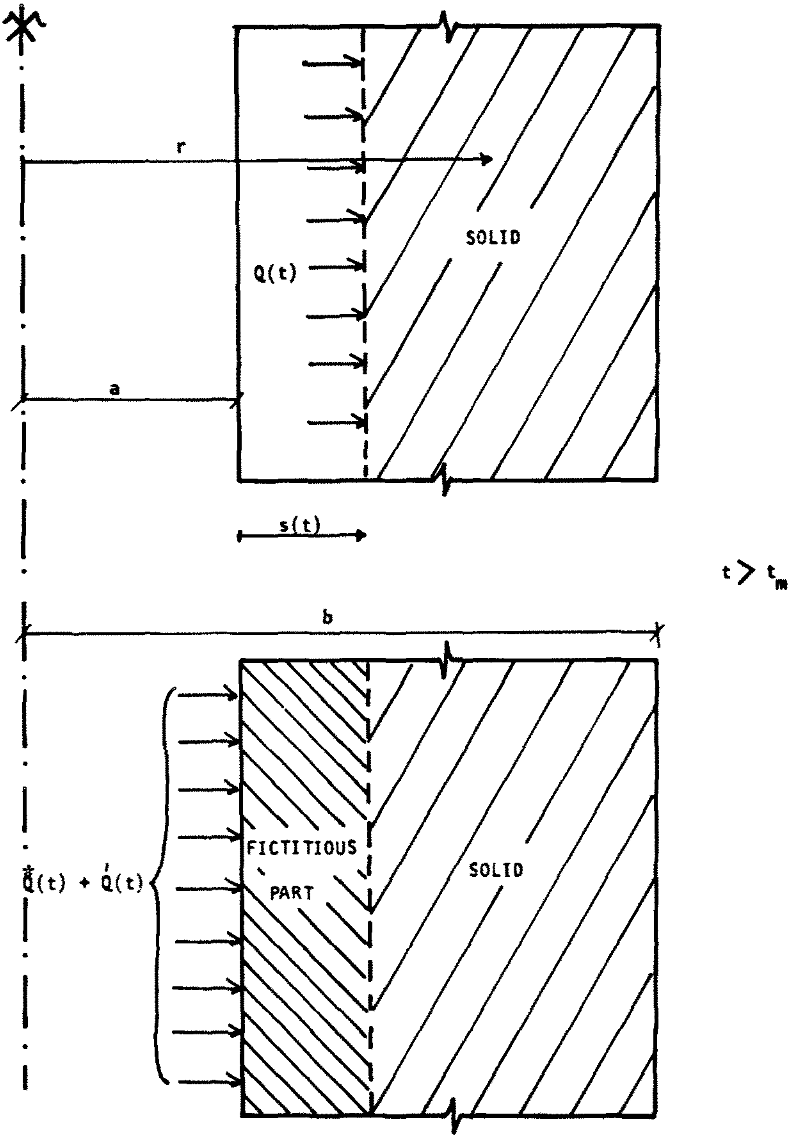

FIG. 2. Equivalent problems for the case of instantaneous removal of the melted portion.

case; as only a starting solution is sought therefore the effect of the outside boundary is negligible.

$$T_0(r,t) = \frac{2}{k}\left(\frac{Kat}{r}\right)^{\frac{1}{2}}\left\{i\,\mathrm{erfc}\frac{r-a}{2(Kt)^{\frac{1}{2}}} - \frac{(3r+a)}{4ar}(Kt)^{\frac{1}{2}} i^2\,\mathrm{erfc}\frac{r-a}{2(Kt)^{\frac{1}{2}}} + \dots\right\} \tag{7}$$

for small values of Kt/a^2.

The temperature distribution in the body after melting is

$$T = T' + T^* \tag{8}$$

Substitution of equation (8) into the first two of equations (6) gives the following two simultaneous integro-differential equations for the two unknowns $s(t)$ and $Q'(t)$:

$$\frac{2}{k}\left(\frac{Ka}{r}\right)^{\frac{1}{2}} \int_0^{t-t_m} Q'(t-\tau)\left\{\frac{1}{2\sqrt{\pi}} \frac{\exp\left(-\frac{(\bar{r}-a)^2}{4K(\tau-t_m)}\right)}{(\tau-t_m)^{\frac{1}{2}}} - \frac{(3\bar{r}+a)}{16a\bar{r}} K^{\frac{1}{2}} \operatorname{erfc}\frac{(\bar{r}-a)}{[K(\tau-t_m)]^{\frac{1}{2}}}\right\} d\tau \tag{9}$$

$$+T_m \sum_{n=0}^{\infty} \sum_{i=0}^{n} a_n y^i (R-1)^{n-i} = T_m$$

$$2(Ka)^{\frac{1}{2}} \int_0^{t-t_m} Q'(t-\tau)\left\{\frac{\bar{r}-a}{4\sqrt{\pi}(\tau-t_m)^{\frac{1}{2}}} \frac{\exp\left(-\frac{(\bar{r}-a)^2}{4K(\tau-t_m)}\right)}{\bar{r}^{\frac{1}{2}}}\left(\frac{3}{4a\bar{r}} - \frac{1}{K(\tau-t_m)}\right)\right.$$

$$\left. + \frac{3}{32}\frac{(\bar{r}+a)}{a\bar{r}^{\frac{5}{2}}} K^{\frac{1}{2}} \operatorname{erfc}\frac{(\bar{r}-a)}{2[K(\tau-t_m)]^{\frac{1}{2}}}\right\} d\tau \tag{10}$$

$$+k\frac{\partial}{\partial r} T_m \sum_{n=0}^{\infty} \sum_{i=0}^{n} a_{ni} y^i (R-1)^{n-i} = -Q^* + \rho l \frac{ds}{dt}$$

where $\bar{r} = a + s(t)$ and $s(t_m) = 0$.

If the problem is confined to small times, $y \ll 1$, i.e. a starting solution, then the second term of equation (9) in the brackets is small as compared to the first term and therefore can be neglected.

This system of integro-differential equations can be written in terms of dimensionless parameters and simplified under the following assumptions:

(a) The temperature distribution is continuous at the transition surface; therefore $a_{00} = 1$.

(b) The dimensionless position $\xi(y)$ of the moving boundary ($\xi(y) = [s(t)/a]$) is less than unity. Thus the binomial expansion can be used. Multiplication of the double series by the terms which appear in front of the summation signs, after regrouping, yields the final form of this pair of integro-differential equations:

$$\int_0^y \frac{Q'(y-\eta)}{Q_0} \frac{e^{-\xi^2(y)/4K_d\eta}}{\eta^{\frac{1}{2}}} d\eta = -\sum_{n=1}^{\infty} \sum_{i=0}^{n} b_{ni} y^i \xi^{n-i} \tag{11}$$

$$\frac{1}{2\sqrt{(\pi K_d)}} \frac{\xi}{[1+\xi]^{\frac{1}{2}}} \int_0^y \frac{Q'(y-\eta)}{Q_0} \frac{e^{-\xi^2(y)/4K_d\eta}}{\eta^{\frac{3}{2}}} d\eta + \frac{3}{8} K_d[1-2\xi(y)] \int_0^y \frac{Q'(y-\eta)}{Q_0} \operatorname{erfc}\frac{\xi}{2\sqrt{K_d\eta}} d\eta$$

$$= -\frac{Q^*}{Q_0} - \sum_{n=0}^{\infty} \sum_{i=0}^{n} c_{ni} y^i \xi^{n-i} + q\frac{d\xi(y)}{dy} \tag{12}$$

with $\xi(0) = 0$ where

$$\xi(y) = \frac{s(t)}{a}, \qquad q = \frac{\rho l a}{Q_0 t_m}$$

$$\begin{aligned}
b_{00} &= a_{00} \\
b_{10} &= a_{10}+(\tfrac{1}{2}a_{00}-\tfrac{1}{2}) \qquad & b_{11} &= a_{11} \qquad & b_{33} &= a_{33} \\
b_{20} &= a_{20}+\tfrac{1}{2}a_{10} & b_{21} &= a_{21}+\tfrac{1}{2}a_{11} & b_{22} &= a_{22} \\
b_{30} &= a_{30}+\tfrac{1}{2}a_{20} & b_{31} &= a_{31}+\tfrac{1}{2}a_{21} & b_{32} &= a_{32}+\tfrac{1}{2}a_{22}
\end{aligned}$$

and

$$\begin{aligned}
c_{00} &= ma_{00} = -q_b \\
c_{10} &= 2ma_{20} & c_{11} &= ma_{21} \\
c_{20} &= 3ma_{30}-\frac{3}{8\sqrt{\pi}}a_{00} & c_{21} &= 2ma_{31}-\frac{3}{8\sqrt{\pi}}a_{11} & c_{22} &= ma_{32} \\
c_{30} &= 4ma_{40}-\frac{3}{8\sqrt{\pi}}a_{20}+\frac{3}{8\sqrt{\pi}}a_{10} & c_{31} &= 3ma_{41}-\frac{3}{8\sqrt{\pi}}a_{21}+\frac{3}{8\sqrt{\pi}}a_{11} \\
c_{32} &= 2ma_{42}-\frac{3}{8\sqrt{\pi}}a_{22} & c_{33} &= ma_{43}.
\end{aligned}$$

There are two unknowns in equations (11) and (12), namely $Q'(y)/Q_0$ and $\xi(y)$, the latter being subject to the condition $\xi(0) = 0$.

These equations are solved for the case of the starting solution, which implies that the leading term of the solution for $\xi(y)$ will be obtained for the case of an arbitrary prescribed $Q^*(y)$ after melting. Equation (12) shows that four terms give a contribution to the rate $d\xi/dy$ at which the melting front advances, that is

$$\xi(y) = \xi_1+\xi_2+\xi_3+\xi_4 \tag{13}$$

where

$$\frac{d\xi_1}{dy} = \frac{1}{q}\left[\frac{Q^*}{Q_0}+c_{00}\right] \tag{14}$$

$$\frac{d\xi_2}{dy} = \frac{1}{q}\sum_{n=1}^{\infty}\sum_{i=0}^{n} c_{ni}y^i\xi_2^{n-i} \tag{15}$$

$$\frac{d\xi_3}{dy} = -\frac{1}{q}\frac{1}{2\sqrt{(\pi K_d)}}\frac{\xi_3}{[1+\xi_3]^{\frac{1}{2}}}\int_0^y \frac{Q'(y-\eta)}{Q_0}\frac{\exp\left(-\dfrac{\xi_3^2}{4K_d\eta}\right)}{\eta^{\frac{3}{2}}}\,d\eta \tag{16}$$

$$\frac{d\xi_4}{dy} = \frac{1}{q}\frac{3}{8}K_d[1-2\xi_4]\int_0^y \frac{Q'(y-\eta)}{Q_0}\,\mathrm{erfc}\left\{\frac{\xi_4}{2\sqrt{(K_d\eta)}}\right\}d\eta. \tag{17}$$

The solution of (14) is

$$\xi_1(y) = \frac{1}{q}\int_0^y\left[\frac{Q^*}{Q_0}+c_{00}\right]d\eta. \tag{18}$$

If $Q^*(y)$ is equal to Q_b after the melting then equation (14) yields the trivial solution $\xi_1(y) = 0$.

In order to evaluate $\xi_2(y)$, it is assumed that the leading non-zero term of the summation corresponds to some specific values of n and i, so that

$$\frac{d\xi_2}{dy} = \frac{1}{q} c_{ni} y^i \xi_2^{n-i}.$$

After the integration

$$\xi_2^{i-n+1} = \frac{1}{q} c_{ni} \frac{y^{i+1}}{i+1} + C$$

where C is an integration constant. The condition $\xi(0) = 0$ can be satisfied only if $i-n+1 > 0$, and since the range of indices in the double summation (15) requires $0 \leq i \leq n$ the only possibility which satisfies this assumption is $n-i = 0$.

Thus

$$\xi_2(y) = \frac{1}{q} \frac{c_{nn}}{n+1} y^{n+1} \tag{19}$$

where n is the smallest integer for which $c_{nn} \neq 0$. The evaluation of $\xi_3(y)$ and $\xi_4(y)$ requires a knowledge of $Q'(y)$ which must be obtained from the other interface condition (11).

$$\int_0^y \frac{Q'(y-\eta)}{Q_0} \frac{\exp\left(-\frac{\xi_{3,4}^2}{4K_d\eta}\right)}{\eta^{\frac{1}{2}}} d\eta = -\sum_{n=1}^{\infty} \sum_{i=0}^{n} b_{ni} y^i \xi_{3,4}^{n-i} \tag{20}$$

where the use of subscripts as in $\xi_{3,4}$, stands for either ξ_3 or ξ_4. In order to obtain $\xi_{3,4}$ it is assumed that the leading non-zero term on the right hand side of equation (20) corresponds to some specific values of n and i. The resulting integral in the equation is simplified for short times in Appendix of Ref. [6] (A.11) with $n = \frac{1}{2}$ yielding

$$\int_0^y \frac{Q'(y-\eta)}{Q_0} \frac{dy}{\eta^{\frac{1}{2}}} = -b_{ni} y^i \xi_{3,4}^{n-i} \tag{21}$$

provided that the condition (A.4) of Ref. [6] holds, equation (21) is a special case of Abel's integral equation with the solution given in Ref. [7]. Thus

$$\frac{Q'(y)}{Q_0} = b_{ni} \frac{1}{\pi} \frac{d}{dy} \int_0^y (y-\eta)^i [\xi_{3,4}(y-\eta)]^{n-i} \frac{dy}{\eta^{\frac{1}{2}}} \tag{22}$$

Equation (16) can be simplified for short times by means of equation (A.11) of Ref. [6], with $n = \frac{3}{2}$.

Therefore

$$\frac{Q'(y)}{Q_0} = -q[1 + \xi_3(y)]^{\frac{1}{2}} \frac{d\xi_3}{dy}. \tag{23}$$

Substitution of (23) into (22) and ignoring the value of $\frac{1}{2}\xi_3(y)$ with respect to unity gives the following equation for $\xi_3(y)$:

$$\xi_3 = \frac{b_{ni}}{\pi q} \int_0^y (y-\eta)^i [\xi_3(y-\eta)]^{n-i} \frac{dy}{\eta^{\frac{1}{2}}}. \tag{24}$$

Substitution of $u = \eta/y$ and $du = d\eta/y$ with the aid of equation (A.6a) and Ref. [6] and using the following requirements: $1-n+i > 0$; $0 \le n-i \le n$, which yield $n = i$, the final form of this equation with the aid of equations (A.12a) and (A.13) of Ref. [6] leads to the desired result

$$\xi_3(y) = \frac{b_{nn}}{\pi q} 2^{2n+1} \frac{(n!)^2}{(2n+1)!} \qquad n \ge 1 \tag{25}$$

where n is the lowest integer for which $b_{nn} \ne 0$.

Comparison of equations (17) and (20) suggests that the complementary error function in equation (17) can be taken as unity. Substitution of $Q'(y)/Q_0$ from equation (22) into equation (17) gives

$$\frac{d\xi_4}{dy} = \frac{1}{q}\frac{3}{8}K_d[1-2\xi_4]\int_0^y -\frac{b_{ni}}{\pi}\left\{\frac{d}{d(y-\eta)}\int_0^{y-n}(y-\eta-\zeta)^i[\xi_4(y-\eta-\zeta)]^{n-i}\frac{d\zeta}{\zeta^{\frac{1}{2}}}\right\}. \tag{26}$$

The following transformation with its limits of integration aids in the integration of this equation:

$$\zeta-\eta = V \qquad -d\eta = dV \qquad \eta = y \qquad V = 0$$
$$\eta = 0 \qquad V = y.$$

Introducing a new parameter $u = y/\eta$ produces an equation similar to equation (24). Then a similar procedure to the one used on equation (24) leads to the following result:

$$\xi_4 = \frac{-b_{nn}}{\pi}\frac{1}{q}\frac{3}{8}K_d\frac{2^{2n+1}(n!)^2}{(n+\frac{3}{2})(2n+1)!}y^{n+\frac{3}{2}} \tag{27}$$

where $n \ge 1$, $y < 1$.

$Q'(y)/Q_0$ is computed from equation (22) subject to the condition $n = i$.

$$\frac{Q'(y)}{Q_0} = -\frac{b_{nn}}{\pi}\frac{d}{dy}\int_0^y (y-\eta)\frac{d\eta}{\eta^{\frac{1}{2}}}. \tag{28}$$

If $n = 1$ then

$$\frac{Q'(y)}{Q_0} = -2\frac{b_{11}}{\pi}y^{\frac{1}{2}}. \tag{29}$$

The final temperature distribution in the body is

$$\theta = \theta' + \theta_b = \theta' + \theta^* \tag{30}$$

where

$$\theta' = -\frac{4b_{11}}{R^{\frac{1}{2}}}y\left[i^2\,\mathrm{erfc}\left\{\frac{R-1}{2\sqrt{(K_d y)}}\right\}\right] \tag{31}$$

for $n = 1$.

The solution $\theta(r, y)$ is valid for

$$1 \gg y \ge 0 \quad \text{and} \quad R \ge 1 + \xi(y)$$

where $\zeta(y)$ can be obtained from equation (13). Thus the starting solution of the problem is obtained.

To extend the range of validity of this starting solution it is now assumed that the following forms can be written:

$$\frac{Q'(y)}{Q_0} = \beta_0 y^{\frac{1}{2}} + \beta_1 y + \beta_2 y^{\frac{3}{2}} + \beta_3 y^2 \tag{32}$$

$$\xi(y) = \alpha_0 y^{\frac{3}{2}} + \alpha_1 y^2 + \alpha_2 y^{\frac{5}{2}} + \alpha_3 y^3 \tag{33}$$

and β_0 and α_0 are taken from the starting solution:

$$\alpha_0 = \frac{4b_{11}}{\pi}$$

$$\beta_0 = -2\frac{b_{11}}{\pi}.$$

A more general heat input of the inner surface of the cylinder is assumed:

$$\frac{Q^*(y)}{Q_0} = \frac{Q_b}{Q_0} + \gamma y^{\frac{1}{2}} + \delta y + \sigma y^{\frac{3}{2}} + \Omega y^2. \tag{34}$$

Substitution of equations (32), (33) and (34) into the dimensionless forms of equations (9) and (10) with the aid of equation (A.3a) of Ref. [6] and expanding the error function in a power series and by equating coefficients of like terms, all terms in the equations can be accounted for.

$$\beta_0 = -\frac{2}{\pi}b_{11}$$

$$\alpha_0 = \frac{2}{3q}\left[\gamma + \frac{2}{\pi}b_{11}\right]$$

$$\beta_1 = -\left[\sqrt{\left(\frac{K_d}{\pi}\right)}b_{11} + \frac{3}{4}b_{10}\alpha_0\right]$$

$$\alpha_1 = \frac{1}{2q}(c_{11} + \delta - \beta_1)$$

$$\beta_2 = -\frac{8}{3\pi}\left[\frac{2}{\sqrt{(K_d\pi)}}b_{11}\alpha_0 - \frac{\sqrt{(\pi K_d)}}{2}\beta_1 + \alpha_1 b_{10} + b_{22}\right]$$

$$\alpha_2 = \frac{2}{5q}\left[\sigma + c_{10}\alpha_0 + \frac{1}{2}\beta_0 K_d + \frac{\beta_0\alpha_0}{2}\sqrt{\left(\frac{\pi}{K_d}\right)} - \beta_2\right]$$

$$\beta_3 = -\frac{15}{16}\left[\frac{1}{4}\pi\beta_0\alpha_0 - \frac{2}{5}\sqrt{(\pi K_d)}\beta_2 + b_{10}\alpha_2 + b_{21}\alpha_0 - \sqrt{\left(\frac{\pi}{K_d}\right)}\beta_0\alpha_1 - \sqrt{\left(\frac{\pi}{K_d}\right)}\beta_1\alpha_0\right]$$

$$\alpha_3 = \frac{1}{3q}\left[\frac{1}{2}\sqrt{\left(\frac{\pi}{K_d}\right)}\alpha_1\beta_0 + \frac{1}{2}\alpha_0\beta_0 + 2\frac{\beta_1\alpha_0}{\sqrt{(\pi K_d)}} - \beta_3 + \frac{3}{8}K_d\beta_1 + \Omega + c_{10}\alpha_1 + c_{22}\right].$$

The final temperature distribution in the hollow cylinder for the case of $y \geq 0$ can be written with the aid of equation (A.3a) of Ref. [6] as

$$\theta(R, y) = \theta^*(R, y) + \frac{\sqrt{\pi}}{R^{\frac{1}{2}}} \sum_{p=1}^{4} 2^{p+1}\Gamma\left(\frac{p}{2}+1\right)\beta_{p-1} y^{(p+1)/2} \operatorname{erfc}\left\{\frac{R-1}{2\sqrt{(K_d y)}}\right\}. \tag{35}$$

3. RESULTS AND REMARKS

Results are given for two different materials with their physical constants and related dimensionless parameters given in Table 1.

Curves showing the motion of the advancing inner surface are computed from equation (33), with respect to $\xi(y)$ and y, and are presented in Figs. 3 and 4.

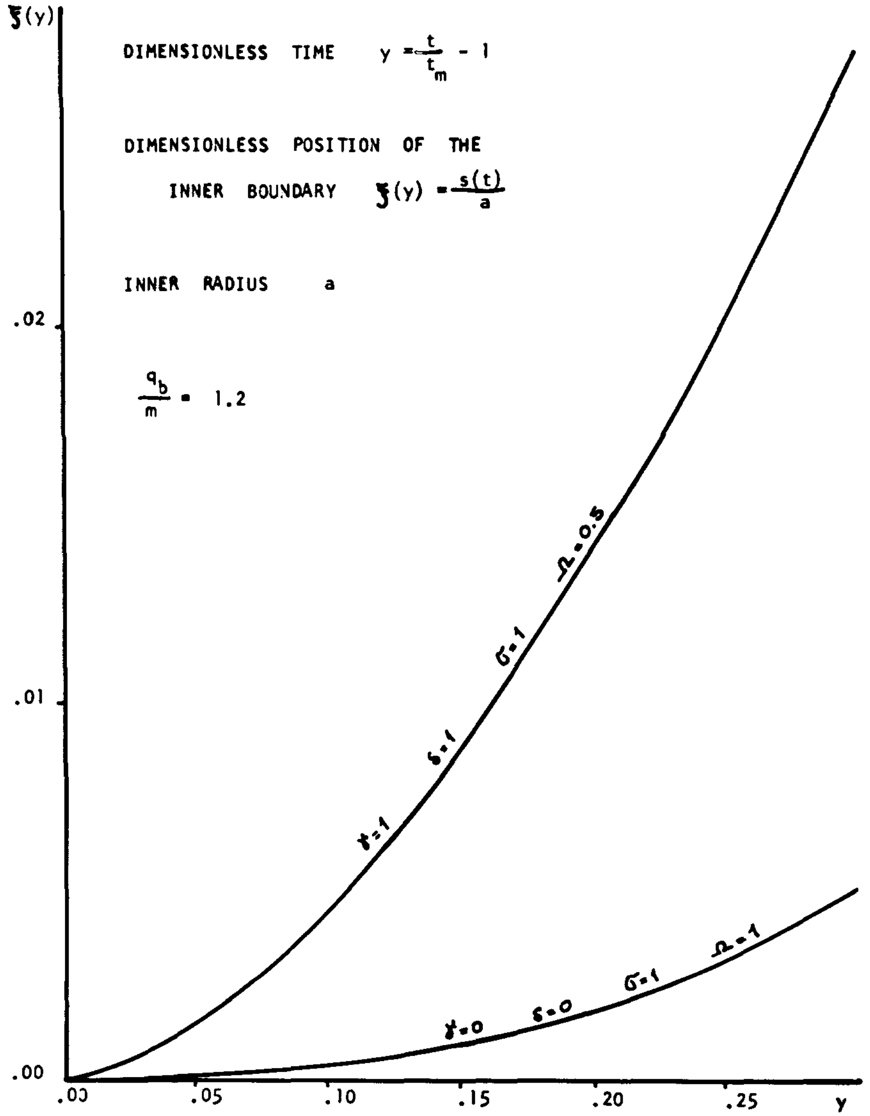

FIG. 3. Advancing inner boundary for ice.

TABLE 1. MATERIALS CONSTANTS [8]

	Ice	Lead
Density (g./cm^3)	0·998	11·34
Latent heat (cal/g)	79·71	6·3
Specific heat (cal/g/°C)	1·008	0·031
Thermal conductivity (cal/cm/°C)	0·004	0·083
Diffusivity (cm^2/sec)	0·00397	0·235
T_m (°C)	100	100
t_m (sec)	1307·0	386·92
q	5·866	3·590
q_b/m	1·2	1
m	0·043	0·179
K_d	0·0058	0·101
a (cm)	30	30
b (cm)	90	90
Q_0	0·311	1·542

The temperature distributions in the vicinity of the inner boundary are plotted for equation (35) for several values of time and are presented in Figs. 5 and 6.

Remarks

1. The starting solution of the position of the advancing boundary can be computed from equation (13). This is written in the general dimensionless form:

$$\xi(y) = c_1 y^{\frac{1}{2}} - c_2 y^{\frac{5}{2}}$$

where c_1 and c_2 are constants and the minus sign of c_2 is due to the curvature of the cylinder. This expression can be approximated for small values of y, such as $y \to \varepsilon$, by

$$\lim_{y \to \varepsilon} \xi(y) \approx c_1 y^{\frac{1}{2}} \qquad \varepsilon \ll 1.$$

For the limiting case, the temperature distribution in the hollow cylinder is very close to that of the starting solution of the finite slab case given in equation (32) of Ref. [6] by Boley. Another way to obtain the slab equations from equations (19) and (10) is to obtain their limiting values as r and a approach infinity.

2. In the temperature distribution, Figs. 5 and 6, the value $\theta = 1·0$ corresponds to the position of the advancing boundary. When the position of the advancing boundary, as predicted by these temperature distribution curves, is compared to that position predicted by equation (33) (Figs. 3 and 4) the results show a good agreement up to $y = 0·2$. The results show some discrepancy for $y > 0·2$.

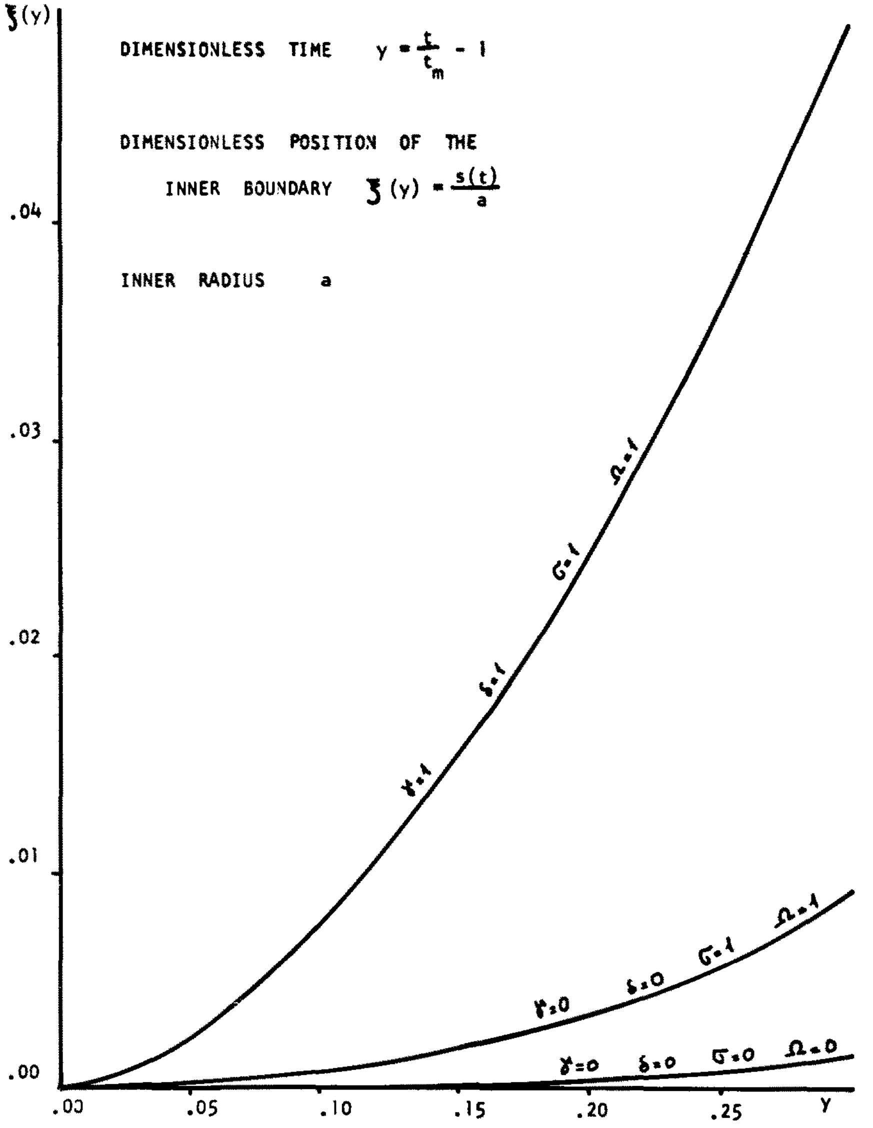

FIG. 4. Advancing inner boundary for lead.

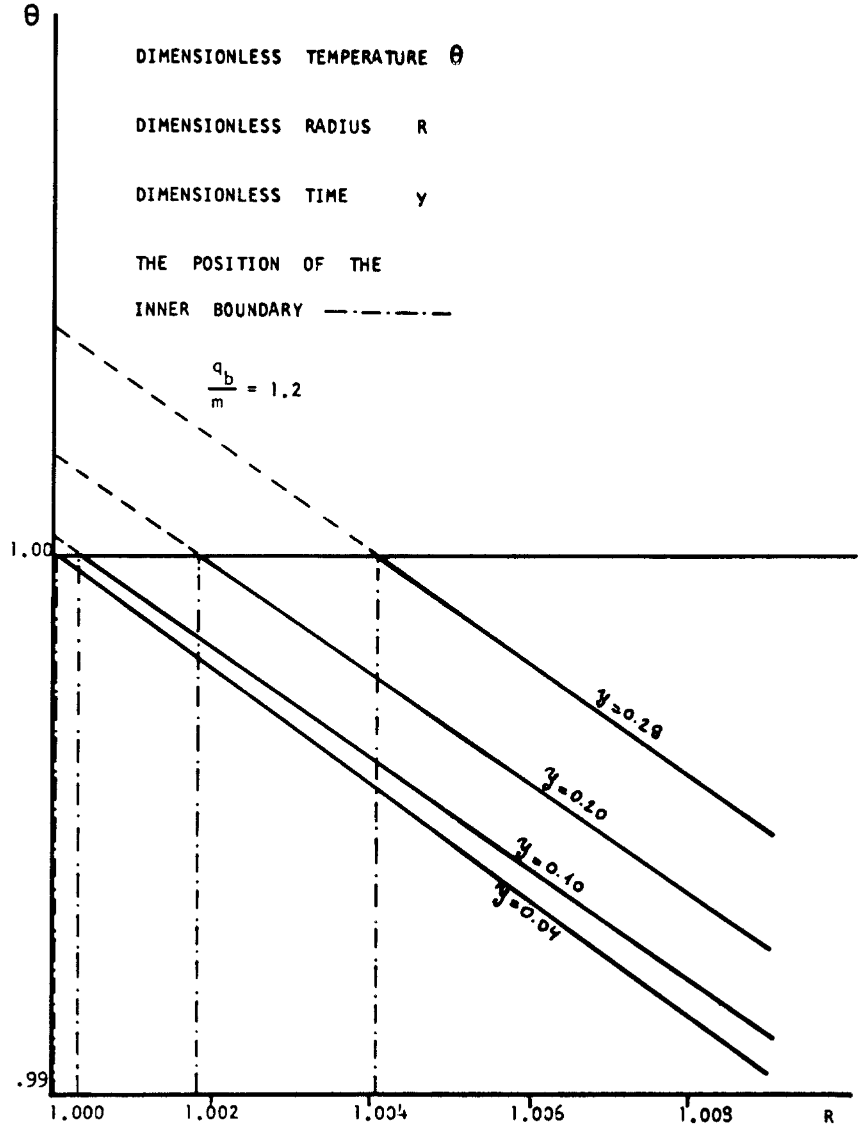

FIG. 5. Temperature distribution after melting for ice.

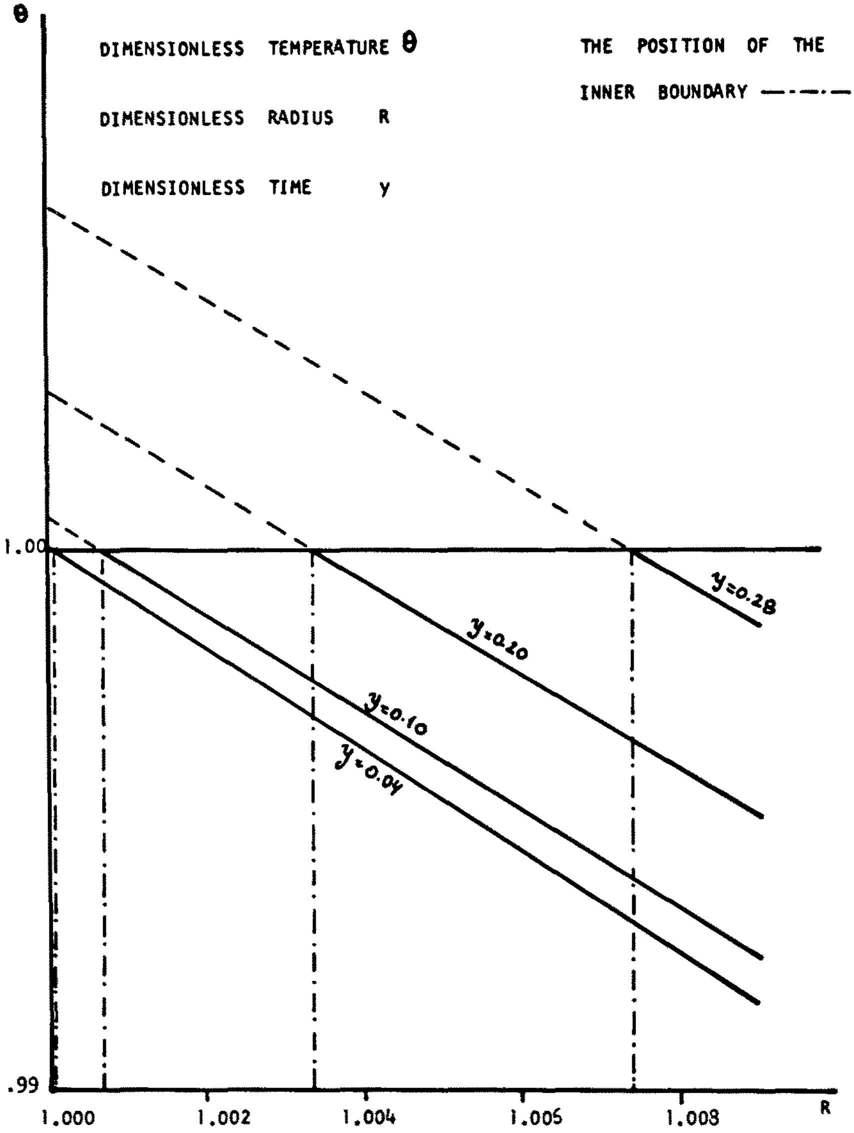

FIG. 6. Temperature distribution after melting for lead.

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Абстракт—Используется, выведенный Болейем, "метод вложения", для случая пустого цилиндра с уменьшающимся внутренным краем, с целью получения общего решения в начальной период времени и для которого периода времени. Для льда и свинца получаются численные результаты, определяющие движение уменьшающегося края и распределение температуры в соседстве движущейгося края.